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When and why is the random force in Brownian motion a Gaussian process

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Abstract

It is shown that the assumptions of causality and time-reversal invariance severely resrict the possibility to describe the fluctuations of a variable in a non-linear Markovian system using a Langevin equation. In fact a theorem is proven which implies that with the aforementioned assumptions a Langevin force which is independent of the state of the system is necessarily Gaussian and white. The theorem furthermore implies that such a description is only possible if the socalled "systematic force" is proportional to the derivative of the logarithm of the equilibrium distribution of the variable. Our analysis is given for a system with one variable which may be either even or odd under time reversal.

Keywords: Brownian motion; Non-linear Markovian system; Langevin equation; Gaussian white noise

1. Introduction

A very successful method to describe fluctuations in a system is the Langevin approach. This approach was first used by Langevin in the description of the stochastic motion of a Brownian particle [1–4] with a time-dependent momentum $p(t)$. In this description the stochastic equation of motion is

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = -\beta p(t) + f(t) \tag{1}$$

where β is the friction coefficient and $\beta p(t)$ is the systematic force. Both this systematic force and the random or Langevin force $f(t)$ are due to collisions with the particles of the fluid in which the Brownian particle is suspended. To use the above stochastic differential equation one must also specify the properties of the random force. Traditional the following two assumptions are made:

(i) The Langevin force is independent of the momentum $p(t)$ of the Brownian particle and has a zero mean value

$$\langle f(t) \rangle = 0 \tag{2}$$

The angled brackets $\langle \dots \rangle$ denote an average over the equilibrium ensemble for the Brownian particle and the particles of the fluid.

(ii) The Langevin force is Gaussian and white with

$$\langle f(t) f(t') \rangle = \alpha \delta(t - t') \tag{3}$$

The stationary nature of the problem then allows one to identify [5]

$$\alpha = 2\beta \langle p^2 \rangle \tag{4}$$

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This relation represents a fluctuation–dissipation theorem.

The above relation (3) implies that

$$\langle p(t) f(t') \rangle = 0 \qquad \text{if } t' > t \tag{5}$$

This is a causality condition for the process described by the Langevin equation (1). It is interesting to note that eq. (3) conversely follows from this causality condition. Thus in order to assure the validity of eq. (3) one may alternatively assume the causality relation (5) to hold.

Given the success and the relative simplicity of the Langevin approach on has in more recent years used this method in a large number of problems. For systems in which the non-fluctuating equation of motion is linear such an extension is straightforward [6,7]. If the equation of motion is non-linear the properties of the random force are no longer easy to establish, however [8,9]. A very lucid discussion of the pitfalls of such an approach is given by Van Kampen in his monograph on "Stochastic Processes in Physics and Chemistry" [4]. As we discussed in a recent paper [10] it is not difficult to formulate a general causality relation in a non-linear system. For the linear Langevin equation we have seen that such a condition may be used as an alternative assumption to determine a property of the random force. The purpose of our paper was to show that for a Langevin force which is independent of the state of the system this general causality condition together with microscopic reversibility imply that the traditional assumption of Gaussian white noise still holds, even in a non-linear system. Furthermore the general causality condition together with microscopic reversibility are only consistent with the nonlinear Langevin equation for a random force which is independent of the state of the system if the systematic part of the force is proportional to the derivative of the logarithm of the equilibrium distribution with respect to the variable. The analysis in our paper was restricted to a one variable system. This restriction is not essential, however.

In this paper we will also consider the special case that there is only one variable y, which may either be symmetric or asymmetric under time reversal and we will derive and present the results derived in ref. [10] in a less formal manner. The evolution of the variable is described phenomenologically, i.e. in the absence of fluctuations, by

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -A(y(t)) \tag{6}$$

where A is an arbitrary function. This equation may have one or several (meta)stable solutions. To account for fluctuations we supplement this equation with a Langevin term independent of $y(t)$ in the following way

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -A(y(t)) + L_A(t) \tag{7}$$

If one averages this equation over an equilibrium ensemble the left hand side of this equation gives zero so that

$$\langle L_\mathrm{A}(t) \rangle = \langle A(y(t)) \rangle \tag{8}$$

The average of the phenomenological "force" over the equilibrium distribution may be written in terms of the reduced equilibrium distribution P_0 of the variable y as

$$\langle A(y) \rangle = \int \mathrm{d}y P_0(y) A(y) \tag{9}$$

It is now convenient to define:

$$B(y(t)) \equiv A(y(t)) - \langle A(y) \rangle \qquad \text{and}$$
$$L(t) \equiv L_\mathrm{A}(t) - \langle A(y) \rangle \tag{10}$$

The Langevin equation may then be written in the form

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -B(y(t)) + L(t) \tag{11}$$

and one obviously has

$$\langle B(y(t)) \rangle = 0 \qquad \text{and} \qquad \langle L(t) \rangle = 0 \tag{12}$$

If one averages eq. (11) for a given initial condition $y(t_0) = y_0$, indicated by a subindex y_0, one needs the conditional average of the Langevin force which using causality, cf. ref. [10], is found to be equal to zero:

$$\langle L(t) \rangle_{y_0} = \langle L(t) \rangle = 0 \qquad \text{for } t > t_0 \tag{13}$$

In the next section causality will be discussed in more detail. Averaging eq. (11) for a given initial condition $y(t_0) = y_0$ one then obtains

$$\frac{\mathrm{d}\langle y(t)\rangle_{y0}}{\mathrm{d}t} = -\langle B(y(t))\rangle_{y_0} \tag{14}$$

Since in general $\langle B(y)\rangle y_0 \neq A(\langle y\rangle_{y_0})$, the average of the Langevin equation does not yield the phenomenological equation (6). On the basis of this fact one has sometimes concluded that the Langevin approach has to be rejected if A is a non-linear function of y. As this very difference is the original of the socalled mode-mode coupling contributions to the relaxation of the variable [11], however, this approach has shown its usefullness.

In this paper we will base our analysis on the Langevin equation for the non-linear case given in eq. (11) together with eq. (12). Though its form may be suggested by the phenomenological equation (6) we will not require eq. (14) to be identical to eq. (6). It is our opinion that for instance the resulting memory effect type contributions to the behaviour of $\langle y(t)\rangle_{y_0}$ due to correlations, though usually small, represent realistic phenomena.

The Langevin equation given in eq. (11) is of course not complete without specification of the properties of the Langevin force $L(t)$. It is often suggested that these properties do not follow from the form of the proposed non-linear Langevin equation [12]. In Sections 2 and 3 of this paper we will show that contrary to this belief the moments of arbitrary order of this Langevin force may be expressed uniquely in equilibrium averages of the product of powers of y with $B(y)$ using only the causal nature of the Langevin equation. It is also found in this context that the Langevin force corresponds to white noise. In Section4 we show that the noise is moreover Gaussian if B is proportional to the derivative of the logarithm of the equilibrium distribution function, i.e. if

$$B(y) \sim \frac{\mathrm{d}}{\mathrm{d}y} \ln P_0(y) \tag{15}$$

In Section 5 we derive the master equation for the general case. The transition probabilities occurring in this equation are given in terms of the moments of the Langevin force. It is shown in Section6 that if one uses furthermore microscopic reversibility the noise is necessarily Gaussian and eq. (15) must hold. The second moment of this noise is found to be given by a fluctuation–dissipation theorem. The generally valid conditions of causality and microscopic reversibility thus severely restrict the possibility to simply add noise to an arbitrary non-linear equation. If for instance eq. (15) is not valid this procedure is not consistent. In the last section a discussion is given of various aspects of the results obtained in this paper.

2. Causality and the Langevin force

We shall now first give a general formulation for the requirement of causality. For *Markov processes* this requirement implies in particular that the variable y at a given time, or any function thereof $g(y)$ at any given time, can not be correlated to the noise at a later time. More generally, if we consider an ordered sequence of n times we demand as condition of causality that

$$\begin{aligned}&\langle g_1(y(t_1))g_2(y(t_2))\dots g_l(y(t_l))\\ &\quad\times L(t_{l+1})L(t_{l+2})\dots L(t_n)\rangle\\ &\quad=\langle g_1(y(t_1))g_2(y(t_2))\dots g_l(y(t_l))\rangle\\ &\qquad\times\langle L(t_{l+1})L(t_{l+2})\dots L(t_n)\rangle\\ &\quad\text{if } t_1\le t_2\le \dots \le t_l < t_{l+1}\le t_{l+2}\le\dots\le t_n\end{aligned} \tag{16}$$

where the functions $g_i(y)$ are arbitrary functions of y. It follows from this causality requirement that moments of the noise satisfy the following product property

$$\begin{aligned}&\langle L(t_1)\dots L(t_n)\rangle\\ &\quad=\langle L(t_1)\dots L(t_l)\rangle\langle L(t_{l+1})\dots L(t_n)\rangle\\ &\quad\text{if } t_i\neq t_j \quad\text{for } 1\le i\le l \quad\text{and } l+1\le j\le n\end{aligned} \tag{17}$$

The proof of this property is given in ref. [10].

The cumulants of the noise are successively defined by the following scheme

$$\langle L(t_1)\rangle \equiv \langle L(T_1)\rangle_c = 0$$

$$\langle L(t_1)L(t_2)\rangle \equiv \langle L(t_1)L(t_2)\rangle_c$$

$$\langle L(t_1)L(t_2)L(t_3)\rangle \equiv \langle L(t_1)L(t_2)L(t_3)\rangle_c$$
$$\begin{aligned}\langle L(t_1)L(t_2)L(t_3)L(t_4)\rangle &\equiv \langle L(t_1)L(t_2)L(t_3)L(t_4)\rangle_c \\ &+ \langle L(t_1)L(t_2)\rangle_c\langle L(t_3)L(t_4)\rangle_c \\ &+ \langle L(t_1)L(t_3)\rangle_c\langle L(t_2)L(t_4)\rangle_c \\ &+ \langle L(t_1)L(t_4)\rangle_c\langle L(t_2)L(t_3)\rangle_c \qquad (18)\end{aligned}$$

where we have used the fact that $\langle L(t)\rangle = 0$. One may of course solve this equation by inversion and express the cumulants in the moments. This is simple to do for the first four cumulants and we will not give these formulae explicitly.

The product property (17) for the moments of the Langevin force given above implies that the cumulants are zero if $t_i \neq t_j$ for any choice of i and j. Correlation functions of any order of the variable may be expressed in terms of integrals over times of the cumulants of the Langevin force which will then only contribute if they are generalized functions. They are in fact given in terms of products of δ-functions in the following way

$$\langle L(t_1)\dots L(t_n)\rangle_c = \alpha_n\delta(t_1-t_n)\delta(t_2-t_n)\dots\delta(t_{n-1}-t_n) \quad \text{for } n\geq 2 \qquad (19)$$

In the next section we will derive explicit expressions for the coefficients α_n.

3. Explicit calculation of the Langevin force cumulants

In order to derive explicit expressions for α_n we use the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[y^n(t)] = -ny^{n-1}(t)B(y(t)) + \left[\frac{y^n(t+\epsilon)-y^n(t-\epsilon)}{y(t+\epsilon)-y(t-\epsilon)}\right]L(t) \qquad (20)$$

where ϵ is infinitessimally small and positive. This equation follows from the Langevin equation for $y(t)$. The first term on the right hand side is self evident in this context. The second term finds its origin in possible discontinuities in $y(t)$ due to the Langevin force which may contain δ-function singularities. By integrating eq. (20) from $(t-\epsilon)$ to $(t+\epsilon)$ one may verify that the prefactor between square brackets is correct, cf ref. 10. It should be emphazised that one should not use either $ny^{n-1}(t-\epsilon)$ or $\frac{1}{2}n[y^{n-1}(t-\epsilon)+y^{n-1}(t-\epsilon)]$ as prefactor. The prefactor is completely determined by the Langevin equation for $y(t)$ and one does not have the freedom to use Itô's or Stratonovich's choices, respectively [8,9].

In the equilibrium ensemble the average of the left hand side of eq. (20) gives zero and we therefore find

$$n\langle y^{n-1}B(y)\rangle = \left\langle\left[\frac{y^n(t+\epsilon)-y^n(t-\epsilon)}{y(t+\epsilon)-y(t-\epsilon)}\right]L(t)\right\rangle \qquad (21)$$

For $n = 1$ this equation reduces to the trivial relation

$$\langle B\rangle = \langle L\rangle = 0 \qquad (22)$$

Integrating the Langevin equation from $(t-\epsilon)$ to $(t+\epsilon)$ it follows that

$$y(t+\epsilon)-y(t-\epsilon) = \int_{t-\epsilon}^{t+\epsilon} L(\tau)\,\mathrm{d}\tau + \mathrm{O}(\epsilon) \qquad (23)$$

This equation may be used to eliminate $y(t+\epsilon)$ in eq. (21). Together with eq. (19) one then obtains, cf. ref. [10],

$$n\langle y^{n-1}B(y)\rangle = \sum_{m=0}^{n-2}\binom{n}{m}\langle y^m\rangle\alpha_{n-m} \quad \text{for } n\geq 2 \qquad (24)$$

By inversion one may obtain the coefficients α_l from this equation in terms of equilibrium correlation functions, cf. ref. [10],

$$\alpha_n = n\langle y^{n-1}B(y)\rangle_c \qquad (25)$$

where the cumulant of a product of n factors, namely $(n-1)$ factors y and one factor $B(y)$, is defined in the usual way. Equation (25) implies that the properties of the Langevin force are completely determined if the equilibrium distribution of the variable y and the function $B(y)$ are known. That this is the case appears to be insufficiently appreciated in discussions of the merits of the non-linear Langevin equation [4].

4. Gaussian noise

We shall now establish the fact that the Langevin force is Gaussian if the function B is related to the equilibrium distribution by

$$B(y) = -\gamma \frac{d}{dy} \ln P_0(y) \tag{26}$$

where γ is a constant. This special case is of interest for two reasons. The first reason is that if the noise is Gaussian the system may equivalently be described by the Fokker–Planck equation. The second reason, as we shall show in Section 6 that microscopic reversibility implies that the noise in the Langevin equation (11) must necessarily be Gaussian.

Multiplying the above equation with y^{n-1} and average one finds the following identity by partial integration

$$\langle y^{n-1}B(y)\rangle = -\gamma \left\langle y^{n-1} \frac{d}{dy} \ln P_0(y) \right\rangle = \gamma(n-1)\langle y^{n-2}\rangle \tag{27}$$

Substitution of this relation into eq. (24) then gives

$$\gamma n(n-1)\langle y^{n-2}\rangle = \sum_{m=0}^{n-2} \binom{n}{m} \langle y^m\rangle \alpha_{n-m} \tag{28}$$

For $n = 2$ this equation becomes

$$\alpha_2 = 2\gamma \tag{29}$$

By induction one finds from eq. (28) that

$$\alpha_n = 2\gamma\delta_{n,2} \tag{30}$$

It follows therefore from causality that the Langevin force is a Gaussian process if $B(y)$ is related to the equilibrium distribution by eq. (26). Thus if e.g. the equilibrium distribution is itself Gaussian, i.e. if

$$P_0(y) \sim \exp\left[-\tfrac{1}{2}\mu y^2\right] \tag{31}$$

and if the systematic force is linear

$$B(y) = \gamma\mu y \tag{32}$$

the Langevin force is not only white but is also necessarily Gaussian.

For a Gaussian random process it is well known [4] that the description using the Langevin equation is equivalent to the description using the Fokker–Planck equation for the time-dependent distribution $P(y, t)$. For the present case this Fokker–Planck equation is given by

$$\begin{aligned}\frac{\partial}{\partial t} P(y, t) &= \gamma \frac{\partial}{\partial y} P(y, t) \frac{\partial}{\partial y} \ln[P(y, t)/P_0(y)] \\ &= \frac{\partial}{\partial y}\left\{ -P(y, t)\gamma \frac{\partial}{\partial y} \ln[P_0(y)] + \gamma \frac{\partial}{\partial y} P(y, t)\right\} \\ &= \frac{\partial}{\partial y}\left\{ B(y)P(y, t) + \gamma \frac{\partial}{\partial y} P(y, t)\right\}\end{aligned} \tag{33}$$

The first term in the last member of this equation contains the systematic force in the Langevin equation $B(y) = -\gamma\, \partial \ln P_0(y)/\partial y$. The second term is due to the Langevin force: thus $\gamma\, \partial \ln P(y, t)/\partial y$ plays the role of the "Langevin force" in the Fokker–Planck equation. In equilibrium the systematic force contribution and the Langevin force contribution to the Fokker–Planck equation clearly cancel each other.

5. The master equation

Returning to the general case we shall now show that the results of Section 3 imply that the stochastic process considered can alternatively be described by a master equation in which the transition probabilities are completely determined by the systematic force $B(y)$ and the equilibrium distribution $P_0(y)$. Consider the following microscopic density distribution

$$w(y, t) \equiv \delta(y - y(t)) \tag{34}$$

where $y(t)$ is a solution of the Langevin equation for a particular realization of the Langevin force and initial condition $y(t=0) = y_0$. The time dependence of $w(y, t)$ is described by

$$\begin{aligned}\frac{\partial}{\partial t} w(y, t) &= \left[\frac{\partial}{\partial y(t)} \delta(y - y(t))\right] \frac{\mathrm{d}}{\mathrm{d}t} y(t) \\ &= \left[-\frac{\partial}{\partial y} \delta(y - y(t))\right][-B(y(t)) + L(t)] \\ &= \left\{\frac{\partial}{\partial y}[B(y(t)) - L(t)]\, \delta(y - y(t))\right\} \\ &= \left\{\frac{\partial}{\partial y}[B(y) - L(t)]\, w(y, t)\right\}\end{aligned} \tag{35}$$

where use has been made of the Langevin equation. Note that in the last member of eq. (35) the force $B(y)$ is a function of y and no longer a function of the time dependent stochastic variable $y(t)$. If one averages over the possible realizations of the Langevin force $B(y)$ is a constant while $B(y(t))$ depends on the realization of the Langevin force.

The conditional probability distribution $P(y_0 \,|\, y, t)$, for which we want to derive a master equation, is obtained from $w(y, t)$ by averaging. The average is over the possible realizations of the Langevin force and a given initial condition $y(t = 0) = y_0$

$$\begin{aligned}P(y_0 \,|\, y, t) &= \langle w(y, t)\rangle_{y_0} \\ &\equiv \langle \delta(y(0) - y_0)\, \delta(y(t) - y)\rangle \\ &\quad / \langle \delta(y(0) - y_0)\rangle\end{aligned} \tag{36}$$

To calculate $P(y_0 \,|\, y, t)$ we use the formal solution of eq. (35) with $w(y, 0) = \delta(y - y_0)$

$$w(y, t) = \left\{\exp \int_0^t \mathrm{d}t' \frac{\partial}{\partial y}[B(y) - L(t')]\right\} \times \delta(y - y_0) \tag{37}$$

It should be emphasized that the differentiation in the exponent not only operates on $B(y)$ but also on $\delta(y - y_0)$. Averaging this equation one obtains

$$P(y_0 \,|\, y, t) = \left\langle \exp \int_0^t \mathrm{d}t' \frac{\partial}{\partial y}[B(y) - L(t')]\right\rangle_{y_0} \times \delta(y - y_0) \tag{38}$$

Equivalently one may write this average as the exponent of the sum of the cumulants

$$P(y_0 \,|\, y, t) = \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n!}\left\langle \left[\int_0^t \mathrm{d}t' \frac{\partial}{\partial y}(B(y) - L(t'))\right]^n \right\rangle_{c, y_0}\right\} \delta(y - y_0) \tag{39}$$

as follows from their definition. For $n = 1$ one has

$$\begin{aligned}&\left\langle \int_0^t \mathrm{d}t' \frac{\partial}{\partial y}(B(y) - L(t'))\right\rangle_{c, y_0} \\ &= \int_0^t \mathrm{d}t' \frac{\partial}{\partial y}(B(y) - \langle L(t')\rangle_{y_0}) \\ &= \int_0^t \mathrm{d}t' \frac{\partial}{\partial y}(B(y) - \langle L(t')\rangle) = t \frac{\partial}{\partial y} B(y)\end{aligned} \tag{40}$$

Notice the fact that due to the causality condition the average of the Langevin force is independent of the initial condition. For $n > 1$ one has, using again the same causality condition,

$$\begin{aligned}&\left\langle \left\{\int_0^t \mathrm{d}t' \frac{\partial}{\partial y}(B(y) - L(t'))\right\}^n \right\rangle_{c, y_0} \\ &= (-1)^n \int_0^t \mathrm{d}t_1 \ldots \mathrm{d}t_n \frac{\partial^n}{\partial y^n} \langle L(t_1) \ldots L(t_n)\rangle_c \\ &= \alpha_n t \frac{\partial^n}{\partial y^n}\end{aligned} \tag{41}$$

Substitution of eqs. (40) and (41) into eq. (39) gives

$$\begin{aligned}&P(y_0 \,|\, y, t) \\ &= \exp\left\{t\left[\frac{\partial}{\partial y} B(y) + \sum_{n=2}^{\infty} \frac{1}{n!}(-1)^n \alpha_n \frac{\partial^n}{\partial y^n}\right]\right\} \\ &\quad \times \delta(y - y_0)\end{aligned} \tag{42}$$

Differentiation of this equation with respect to time yields

$$\frac{\partial}{\partial t}P(y_0\,|\,y,t) = \left[\frac{\partial}{\partial y}B(y) + \sum_{n=2}^{\infty}\frac{1}{n!}(-1)^n\alpha_n\frac{\partial^n}{\partial y^n}\right] \times P(y_0\,|\,y,t) \quad (43)$$

This is simply the Kramers–Moyal expansion of the master equation [4]

$$\frac{\partial}{\partial t}P(y_0\,|\,y,t) = \int \mathrm{d}y'[T(y\,|\,y')P(y_0\,|\,y',t) - T(y'\,|\,y)P(y_0\,|\,y,t)] \quad (44)$$

where the moments of the transition probability $T(y\,|\,y')$ are given by

$$\sigma_1(y) \equiv \int \mathrm{d}y'(y'-y)T(y'\,|\,y) = -B(y) \quad (45)$$

$$\sigma_n(y) \equiv \int \mathrm{d}y'(y'-y)^n T(y'\,|\,y) = \alpha_n = n\langle y^{n-1}B(y)\rangle_c \quad (46)$$

Using generalized functions one thus finds for the transition probability

$$T(y\,|\,y') = \left[\frac{\partial}{\partial y}B(y) + \sum_{n=2}^{\infty}\frac{1}{n!}(-1)^n\alpha_n\frac{\partial^n}{\partial y^n}\right] \times \delta(y-y') \quad (47)$$

which is indeed completely determined in terms of $B(y)$ and the equilibrium distribution. Note the fact that one is a left eigenvector of $T(y'\,|\,y)$ with eigenvalue zero so that the second term on the right hand side of eq. (44) vanishes. For the special case of Gaussian noise discussed in the previous section one has $\alpha_n = 2\gamma\delta_{n,2}$ and the master equation above reduces to the Fokker–Planck equation.

Substituting the equilibrium distribution into the master equation one has

$$\frac{\partial}{\partial y}B(y)P_0(y) + \sum_{n=2}^{\infty}\frac{1}{n!}(-1)^n\alpha_n\frac{\partial^n}{\partial y^n}P_0(y) = 0 \quad (48)$$

Integrating this expression one finds

$$B(y) = -\sum_{n=2}^{\infty}\frac{1}{n!}(-1)^n\alpha_n[P_0(y)]^{-1} \times \frac{\partial^{n-1}}{\partial y^{n-1}}P_0(y) \quad (49)$$

where we have used the fact that $\langle B(y)\rangle = 0$ in order to determine the integration constant. It follows from this equation that if the Langevin force is Gaussian $B(y)$ is given by eq. (26). Together with the results of Section 4 it then follows that the Langevin force is Gaussian *if and only if* $B(y)$ is given by eq. (26).

6. Microscopic time reversibility

From invariance for time reversal on the microscopic level it follows that the transition probabilities satisfy the symmetry relation

$$T(y'\,|\,y)P_0(y) = T(\tau y\,|\,\tau y')P_0(y') \quad (50)$$

where $\tau = 1$ if y is an even variable and $\tau = -1$ if y is an odd variable. For the equilibrium distribution microscopic reversibility implies that

$$P_0(y) = P_0(\tau y) \quad (51)$$

In order to investigate the implications of eq. (50) we introduce the following moments

$$T_{lm} \equiv \int \mathrm{d}y\,\mathrm{d}y'\,y'^l T(y\,|\,y')P_0(y')(y')^m \quad (52)$$

Microscopic time reversal invariance, cf. eq. (50), implies that these moments satisfy the following symmetry relation

$$T_{lm} = \tau^{l+m}T_{ml} \quad (53)$$

The quantities T_{lm} may be expressed in the moments α_n by making use of eqs. (47) and (49), cf. ref. [10],

$$T_{lm} = -l\sum_{n=2}^{l+m}\frac{1}{n}\alpha_n\left\{\binom{l+m-1}{n-1} - \binom{l-1}{n-1}\theta_{l-n}\right\} \times \langle y^{l+m-n}\rangle \quad (54)$$

where $\theta_j \equiv 1$ if $j \geq 0$ and $\theta_j \equiv 0$ if $j < 0$. If this identity is substituted into the symmetry relation eq. (53) one finds

$$l \sum_{n=2}^{l+m} \frac{1}{n} \alpha_n \left\{ \binom{l+m-1}{n-1} - \binom{l-1}{n-1} \theta_{l-n} \right\} \langle y^{l+m-n} \rangle = m \sum_{n=2}^{l+m} \frac{1}{n} \alpha_n \tau^n \left\{ \binom{l+m-1}{n-1} - \binom{m-1}{n-1} \theta_{m-n} \right\} \times \langle y^{l+m-n} \rangle \quad (55)$$

where we used the following property of the equilibrium distribution, cf. eq. (51),

$$\langle y^j \rangle = \tau^j \langle y^j \rangle \quad (56)$$

One may verify the fact that the $n = 2$ contributions on the left and on the right hand side of eq. (55) are identical for all values of l and m. The higher order α_n's must consequently be found from the following equation

$$l \sum_{n=3}^{l+m} \frac{1}{n} \alpha_n \left\{ \binom{l+m-1}{n-1} - \binom{l-1}{n-1} \theta_{l-n} \right\} \langle y^{l+m-n} \rangle = m \sum_{n=3}^{l+m} \frac{1}{n} \alpha_n \tau^n \left\{ \binom{l+m-1}{n-1} - \binom{m-1}{n-1} \theta_{m-n} \right\} \times \langle y^{l+m-n} \rangle \quad (57)$$

Using this equation we one may show, cf. ref. [10], by induction that

$$\alpha_n = 0 \qquad \text{for } n > 2 \quad (58)$$

which implies that the noise is both Gaussian and white.

7. Conclusions

In the preceding sections we have in fact proven the following three theorems for a stochastic differential equation of the form

$$\frac{\partial}{\partial t} y(t) = -B(y(t)) + L(t) \quad (59)$$

where $L(t)$ is independent of the value of the variable and has a zero average value:

(i) It follows from causality that the noise (Langevin force) is white;

(ii) It follows furthermore from causality that the noise is also Gaussian if and only if $B(y)$ is related to the equilibrium distribution by

$$B(y) \sim \frac{\mathrm{d}}{\mathrm{d}y} \ln P_0(y) \quad (60)$$

(iii) It follows from causality and microscopic reversibility that the noise is Gaussian if the variable is either even or odd under time reversal.

The first theorem is not unexpected. It implies that the value of the noise at a given time is uncorrelated to its value at another time. In particular it is shown that the cumulants of products of the noise for a sequence of times, which are not all equal, are zero. The second and the third theorem severely restrict the possibility to describe the fluctuations of a non-linear system of one variable using a Langevin equation of the type given in eq. (59). They imply in fact that this is only possible if the non-linearity is such that it satisfies eq. (60) as long as the very general conditions of causality and microscopic reversibility are assumed to hold. If eq. (60) does not hold, as may be the case if the system is far from equilibrium, it is necessary to use an alternative description for fluctuations. A possible approach is the introduction of socalled multiplicative noise where the Langevin force explicitly depends on the variable [4,13]. It should be emphasised that in our analysis the restriction to a Langevin force which is independent of the variable is crucial for the validity of the above theorems

Finally we note with respect to eq. (60) that it corresponds within the context of nonequilibrium thermodynamics to the usual definition of thermodynamic forces as derivatives of the logarithm of the equilibrium distribution. This thermodynamic force is in fact most properly identified with the phenomenological force $A(y)$ in the introduction. As, however, $\langle \mathrm{d}/\mathrm{d}y \ln P_0(y) \rangle = 0$ it is not necessary to distinguish between $A(y)$ and $B(y)$ in this case, cf. eq. (10).

Though we have considered in the previous sections a system described by only one variable, it is clear that analogous results may be obtained for a system with several variables. We intend to address this more general case in a future publication.

References

1 P. Langevin, C. R. Acad. Sci. (Paris) 146 (1908) 530.
2 G.E. Uhlenbeck and L.S. Ornstein, Phys. Rev. 36 (1930) 823.
3 N. Wax, Selected Papers on Noise and Stochastic Processes (Dover, New York, NY, 1954).
4 N.G. van Kampen, Stochastic Properties in Physics and Chemistry (Elsevier, Amsterdam, 1981).
5 A. Einstein, Ann. Physik (4) 17 (1905) 549; 19 (1906) 371.
6 L. Onsager and S. Machlup, Phys. Rev. 36 (1953) 823.
7 L. Landau and E.M. Lifshitz, Fluid Mechanics (Pergamon Press, New York, NY, 1959).
8 K. Itô, Proc. Imp. Acad. Tokyo 20 (1944) 519; Mem. Am. Math. Soc. 4 (1951) 51.
9 R.L. Stratonovich, SIAM J. Control 4 (1966) 362; Topics in Theory of Random Noise, vol. I, Ch. 4 (Gordon and Breach, New York, NY, 1963).
10 P. Mazur and D. Bedeaux, Preprint, to be published in Physica A
11 See e.g.: S.K. Ma, Modern Theory of Critical Phenomena, Ch. XIII (Benjamin, Reading, MA, 1976).
12 See in particular ref. [4], p. 250.
13 G.L. Eyink, Preprint, Stat. Phys. (1991).